Random dynamical systems and transfer operator cocycles

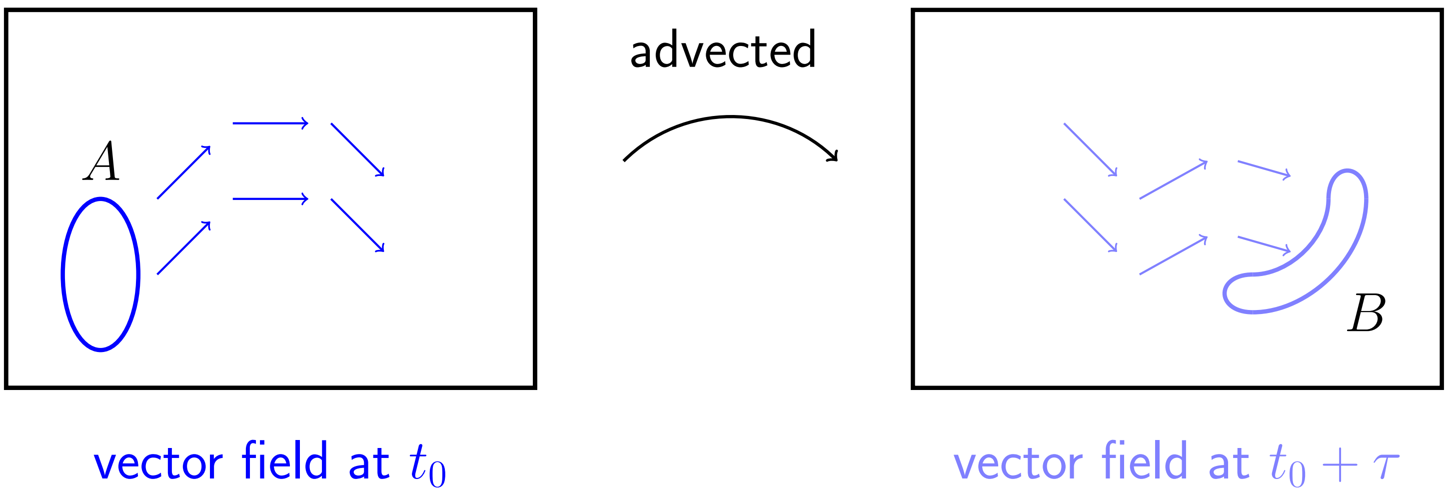
Rua Murray (University of Canterbury, NZ)

joint work with Gary Froyland (UNSW) and Cecilia González-Tokman (UQ)

NZMC — December, 2018

Example 1: mass transport in a nonautonomous flow

- Mass is advected by a time-varying vector field $\mathbf{v}(t, x)$
- Transport equation: $\frac{\partial u}{\partial t} = -\frac{\partial}{\partial \mathbf{x}} \cdot (u\,\mathbf{v})$
- Same initial distribution may evolve differently in time τ depending on initial time!
- Application: global ocean dynamics [Gary Froyland + many others]
- **Problem:** compute transport reliably (v from data, not evolution equation)



Example 2: Random dynamical system

Ingredients

- Phase space X
- Collection $\{f_k\}_{k\in\Xi}$, $f_k : X \to X$
- A sequence $\{\xi_1, \xi_2, \xi_3, \ldots\}$ of Ξ-valued random variables

Random dynamics on X

$$x_t = f_{\xi_t}(x_{t-1})$$

[choose a random map each time and apply it]

Well studied since the 1980s (eg, Ξ finite, $\{\xi_t\}$ IID)

- Expanding interval maps: Pelikan (1984), Morita (1985)
- Fractals, via "Iterated function systems": Hutchinson (1981), Barnsley, ...

Mathematical setup

[Look at discrete time only – discretise DE etc]

Autonomous: phase space X, $f : X \to X$ (map)

in time t: $x \mapsto f^t(x) = \underbrace{f \circ \cdots \circ f}_{t \text{ times}}(x)$

Mathematical setup

[Look at discrete time only – discretise DE etc]

Autonomous: phase space X, $f : X \to X$ (map)

in time t: $$x \mapsto f^t(x) = \underbrace{f \circ \cdots \circ f}_{t \text{ times}}(x)$$

Nonautonomous: the map also depends on initial time,

$$F : T \times X \to T \times X, \qquad F(t_0, x) = (t_0 + 1, f_{t_0}(x))$$

in time t, $$(0, x) \mapsto (t, \underbrace{f_{t-1} \circ \cdots \circ f_1 \circ f_0}_{f_0^{(t)}}(x))$$

Mathematical setup

[Look at discrete time only – discretise DE etc]

Autonomous: phase space X, $f : X \to X$ (map)

in time t: $$x \mapsto f^t(x) = \underbrace{f \circ \cdots \circ f}_{t \text{ times}}(x)$$

Nonautonomous: the map also depends on initial time,

$$F : T \times X \to T \times X, \qquad F(t_0, x) = (t_0 + 1, f_{t_0}(x))$$

in time t, $$(0, x) \mapsto (t, \underbrace{f_{t-1} \circ \cdots \circ f_1 \circ f_0}_{f_0^{(t)}}(x))$$

Random dynamical system: "skew product" over more general base dynamics

Space: $\Omega \times X$

Base dynamics: $\sigma : \Omega \to \Omega$

[ergodic wrt $\mathbb{P}$]

Fibres: $X_\omega := \{\omega\} \times X$

Fibre dynamics: $f_\omega : X_\omega \to X_{\sigma\omega}$

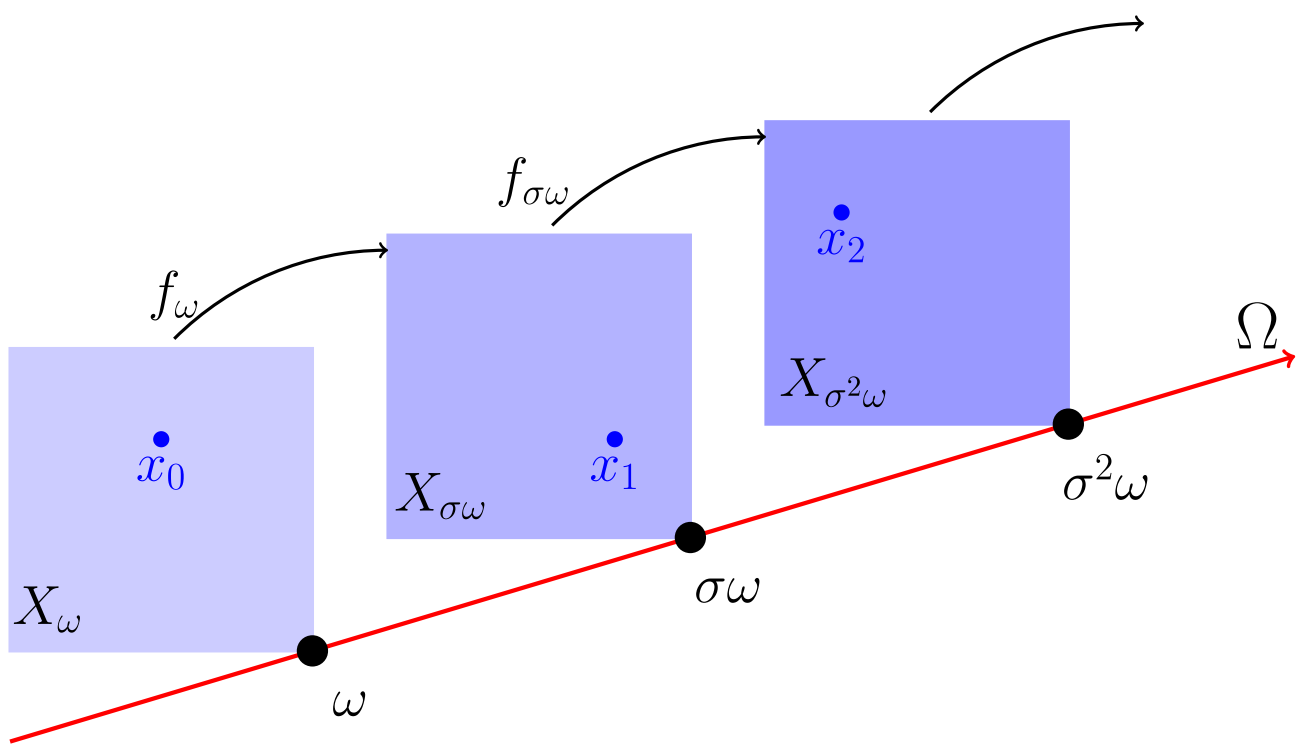
Random dynamics: $x_{t+1} = f_{\sigma^t\omega}(x_t)$

Skew product setup and picture

- Morita attributes the skew-product formulation to Kakatuni (1957)
- the skew-product is an **autonomous dynamical system** on $\Omega \times X$

$$F(\omega, x) = (\sigma\omega, f_\omega(x)), \qquad F^t(\omega, x) = (\sigma^t\omega, f_\omega^{(t)}(x))$$

- the *fibre-to-fibre* mappings induce a **cocycle** on X: $f_\omega^{(t_1+t_2)} = f_{\sigma^{t_1}\omega}^{(t_2)} \circ f_\omega^{(t_1)}$



Skew product for IID random dynamical systems

- suppose ξ_k are IID with each $\xi_k \sim q$ on Ξ
- put $\Omega = \Xi^\infty$, $\mathbb{P} = q^\infty$ and σ the **left shift**

$$[\sigma\omega]_k = \omega_{k+1}$$

- the fibre-map f_ω is chosen according to the first coordinate of ω

Three views with general base dynamics $\sigma : \Omega \to \Omega$

1. An autonomous dynamical system $F : \Omega \times X \circlearrowright$ inducing *fibre-to-fibre* maps
2. A *random dynamical system* on X (use f_ω each time, ω-dynamics hidden)
3. A *Markov chain* on X with transition probabilities

$$pr(x_t \in A \mid x_{t-1} = y) = q(y, A) = \int_\Omega \mathbf{1}_A(f_\omega(y))\, d\mathbb{P}(\omega).$$

Physical measures

In the IID case, all three viewpoints are equivalent.

Morita (1987) *When the base is IID, the F-invariant probability measures are of the form $\mu_1 \times \mathbb{P}$, where μ_1 is an invariant probability for the Markov chain.*

In the IID case, invariant probabilities for the Markov chain can be decomposed into **physical measures**.

Definition. A probability measure μ on X is an **SRB measure** if

$$\lim_{t\to\infty} \frac{1}{t}\left[g(x) + g(f_\omega(x)) + \cdots + g(f_\omega^{(t-1)}(x))\right] = \int_X g\, d\mu$$

for μ–a.e. $x \in X$, $\mathbb{P}$–a.e. ω.

When X is equipped with a *natural measure* m (for example length, volume), an SRB measure is a physical measure when its support has positive m–measure.

Buzzi (2000) *Without IID, certain random expanding interval maps admit finitely many physical measures, arising as X–marginals of F–invariant measures.*

Transfer operators: annealed case

- **Transfer operator**: $\mathcal{P} : L^1(X) \to L^1(X)$ when $f : X \circlearrowleft$

  if x is distributed with density φ, then $f(x)$ is distributed with density $\mathcal{P}\varphi$
- **Random transfer operator**: $\mathcal{P}_\omega : L^1(X) \to L^1(X)$ from $f_\omega : X_\omega \to X_{\sigma\omega}$
- **Annealed transfer operator:** $\overline{\mathcal{P}} = \int_\Omega \mathcal{P}_\omega \, d\mathbb{P}(\omega)$

Easy calculation: *if the Markov chain has an absolutely continuous invariant probability* μ_1 *then its density* $\varphi_1 = \frac{d\mu_1}{dm}$ *satisfies* $\overline{\mathcal{P}}\varphi_1 = \varphi_1$.

Froyland (1999): for IID expanding interval maps the operator $\overline{\mathcal{P}}$ is robust to certain finite-rank Galerkin-type projections, allowing the computation of physical measures.

Transfer operators: quenched case

When the base is not IID, study explicitly

- $f_\omega : X_\omega \to X_{\sigma\omega}$
- associated $\mathcal{P}_\omega : L^1(X_\omega) \to L^1(X_{\sigma\omega})$

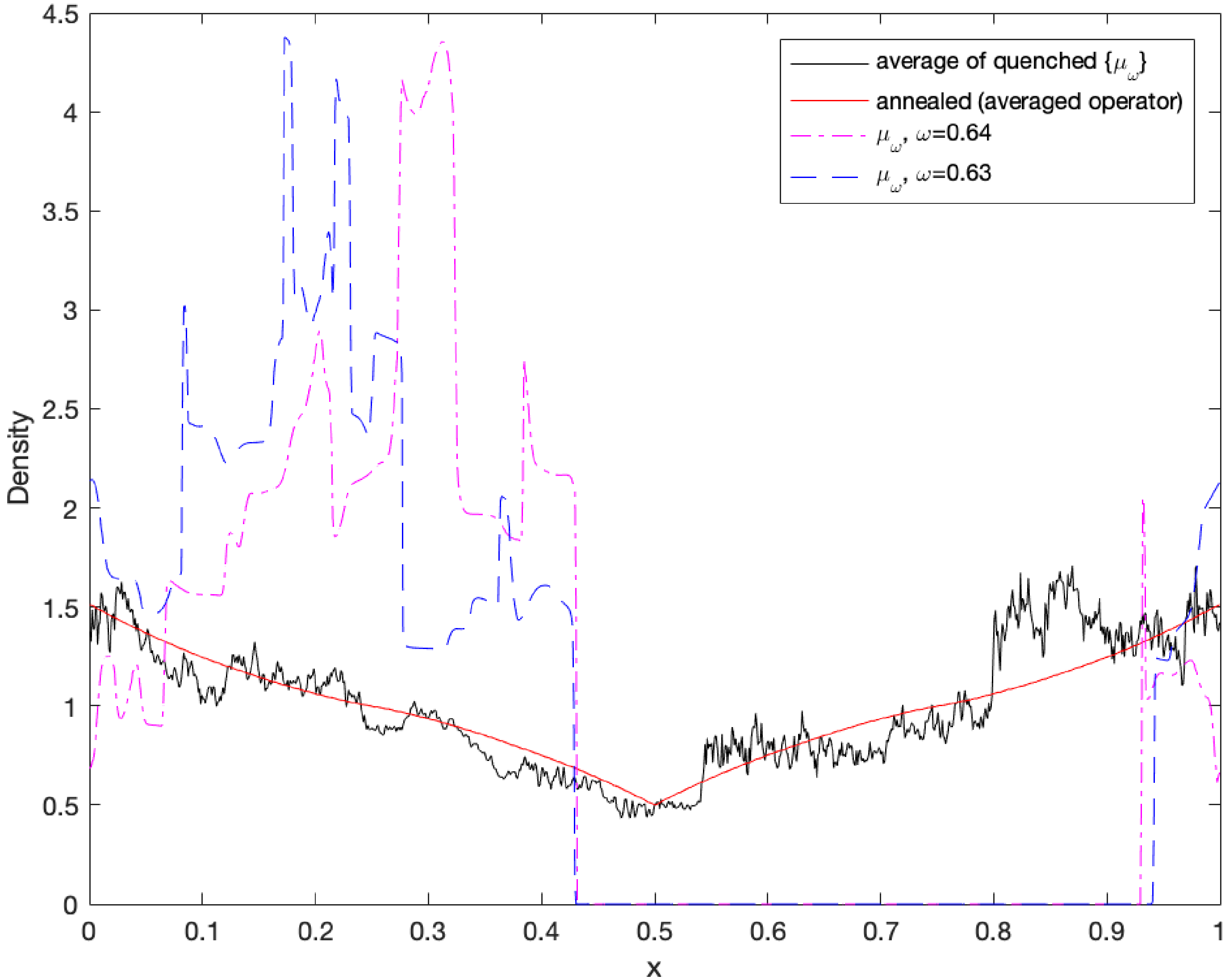
1. F–invariant measures ν disintegrate as $d\nu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$
2. **Mass transfer** is controlled by behaviour of random products

$$\mathcal{P}_\omega^{(t)} := \mathcal{P}_{\sigma^{t-1}\omega} \circ \cdots \circ \mathcal{P}_{\sigma\omega} \circ \mathcal{P}_\omega$$

Goal: instead of obtaining eigenvectors[a] for $\overline{\mathcal{P}}$ obtain Lyapunov exponents and random subspaces for $\{\mathcal{P}_\omega^{(t)}\}$.

[a] Subunit eigenvalues relate to *mixing*.

- **random densities** $\varphi_\omega^* \in L^1(X_\omega)$
- **equivariance**: $\mathcal{P}_\omega \varphi_\omega^* = \varphi_{\sigma\omega}^*$
- **attracting**: $\mathcal{P}_{\sigma^{-t}\omega}^{(t)} \varphi_0 \to \varphi_\omega^*$ as $t \to \infty$
- **physical measure**: $\frac{d\mu}{dm} = \int_\Omega \varphi_\omega^* \, d\mathbb{P}(\omega)$



Densities computed via a discretised transfer operator with $\Omega = S^1$, σ rotation.

Stability of transfer operator cocycles

Combining

- classical on work on Multiplicative Ergodic Theorems for random matrix products
- a framework established by Keller and Liverani in (1998)
- Buzzi's use of random inequalities of the type

$$\|\mathcal{P}_\omega \varphi\|_B \le \alpha(\omega)\, \|\varphi\|_B + K(\omega)\, |\varphi|_{L^1}$$

  in terms of **random constants** with $\int \log \alpha(\omega)\, d\mathbb{P}(\omega) < 0$

Froyland, Gonzalez-Tokman and Quas (2016) have proved a series of **Multiplicative Ergodic Theorems** for transfer operator cocyles, along with robustness of their random Lyapunov subspaces to certain types of perturbations.

Stability of transfer operator cocycles

Combining

- classical on work on Multiplicative Ergodic Theorems for random matrix products
- a framework established by Keller and Liverani in (1998)
- Buzzi's use of random inequalities of the type

$$\|\mathcal{P}_\omega \varphi\|_B \le \alpha(\omega)\, \|\varphi\|_B + K(\omega)\, |\varphi|_{L^1} \qquad (*)$$

  in terms of **random constants** with $\int \log \alpha(\omega)\, d\mathbb{P}(\omega) < 0$

Froyland, Gonzalez-Tokman and Quas (2016) have proved a series of **Multiplicative Ergodic Theorems** for transfer operator cocyles, along with robustness of their random Lyapunov subspaces to certain types of perturbations.

But, applying the results required the (possibly prohibitively expensive) task of calculating $\mathcal{P}_\omega^{(T)}$ for a large T.

After considerable effort:

Suppose $\mathcal{P}_\omega^{(T_0)}$ satisfies a random version of (*), even if $\mathcal{P}_\omega$ does not.

We can perturb $\mathcal{P}_\omega$ at every time step, and

- harvest stability results
- actually calculate via finite rank projections of $\mathcal{P}_\omega$

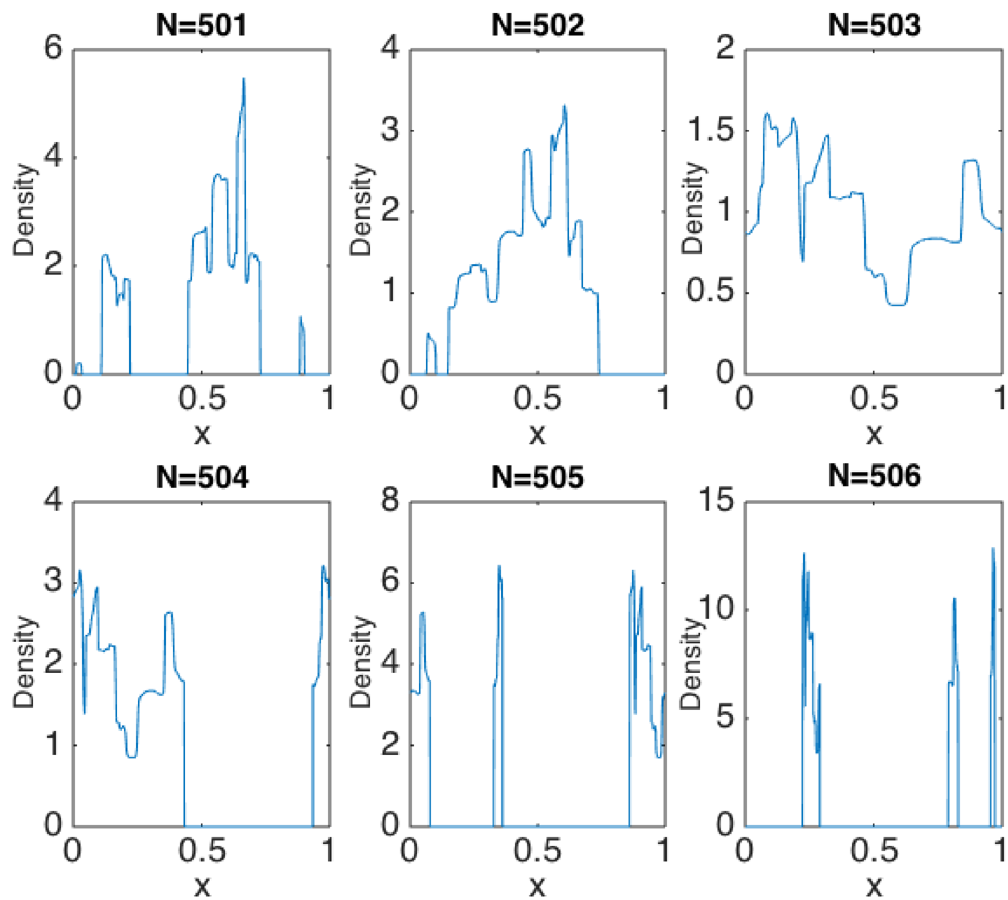
Positive results

- (σ, Ω) ergodic (invertible) process
- $f_\omega : I \to I$
  - finitely many C^2 branches
  - eventually expanding on average: for some N_0

$$\int \log(\min_I |f_\omega^{(N_0)\prime}|) \, d\mathbb{P}(\omega) > 0$$

Example

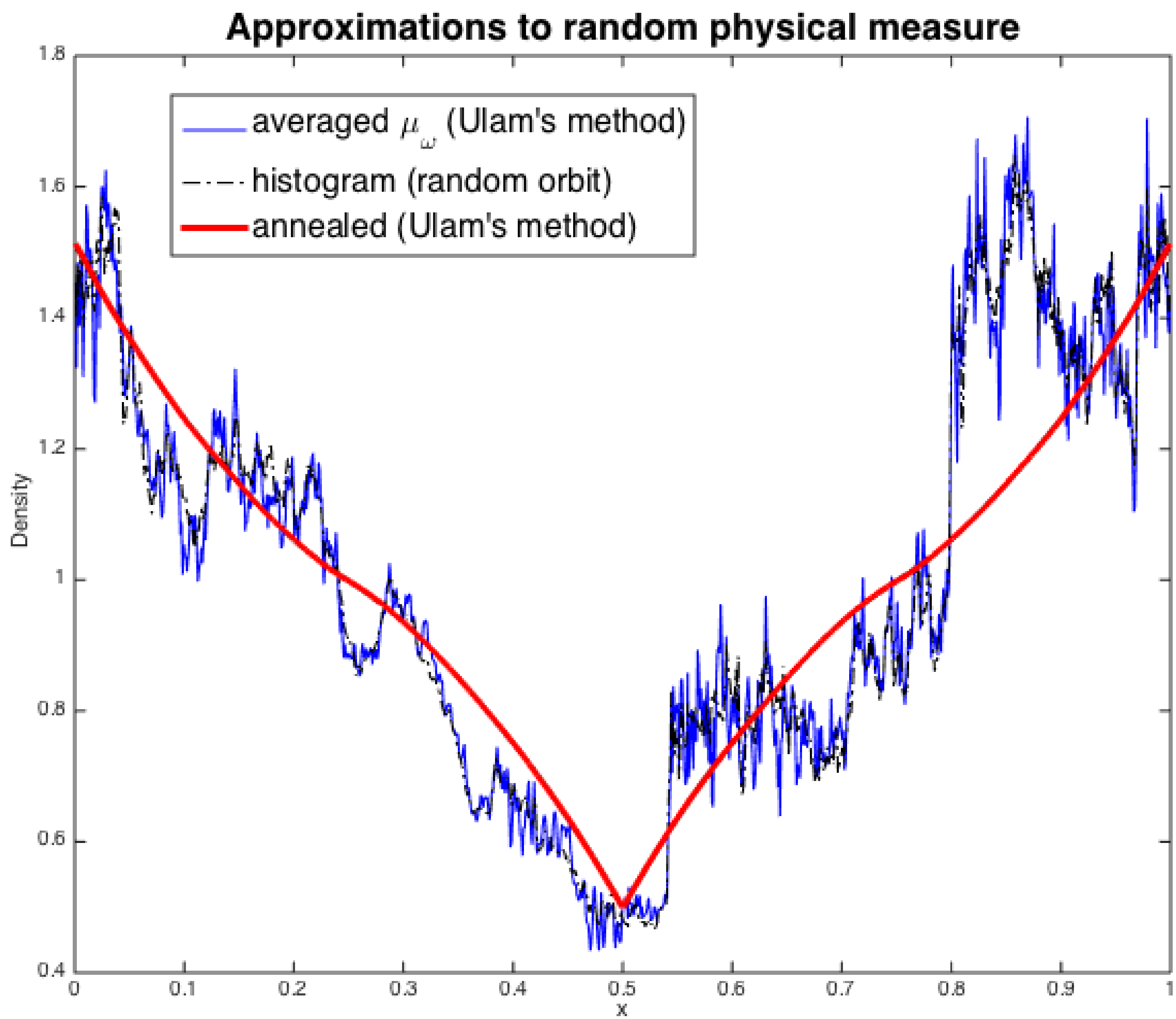
- β–transformations with random shift determined by ω
- slopes $\beta_1 = 2.1$, $\beta_2 = 0.5$ (each occurs 1/2 time)
- σ is irrational rotation on circle



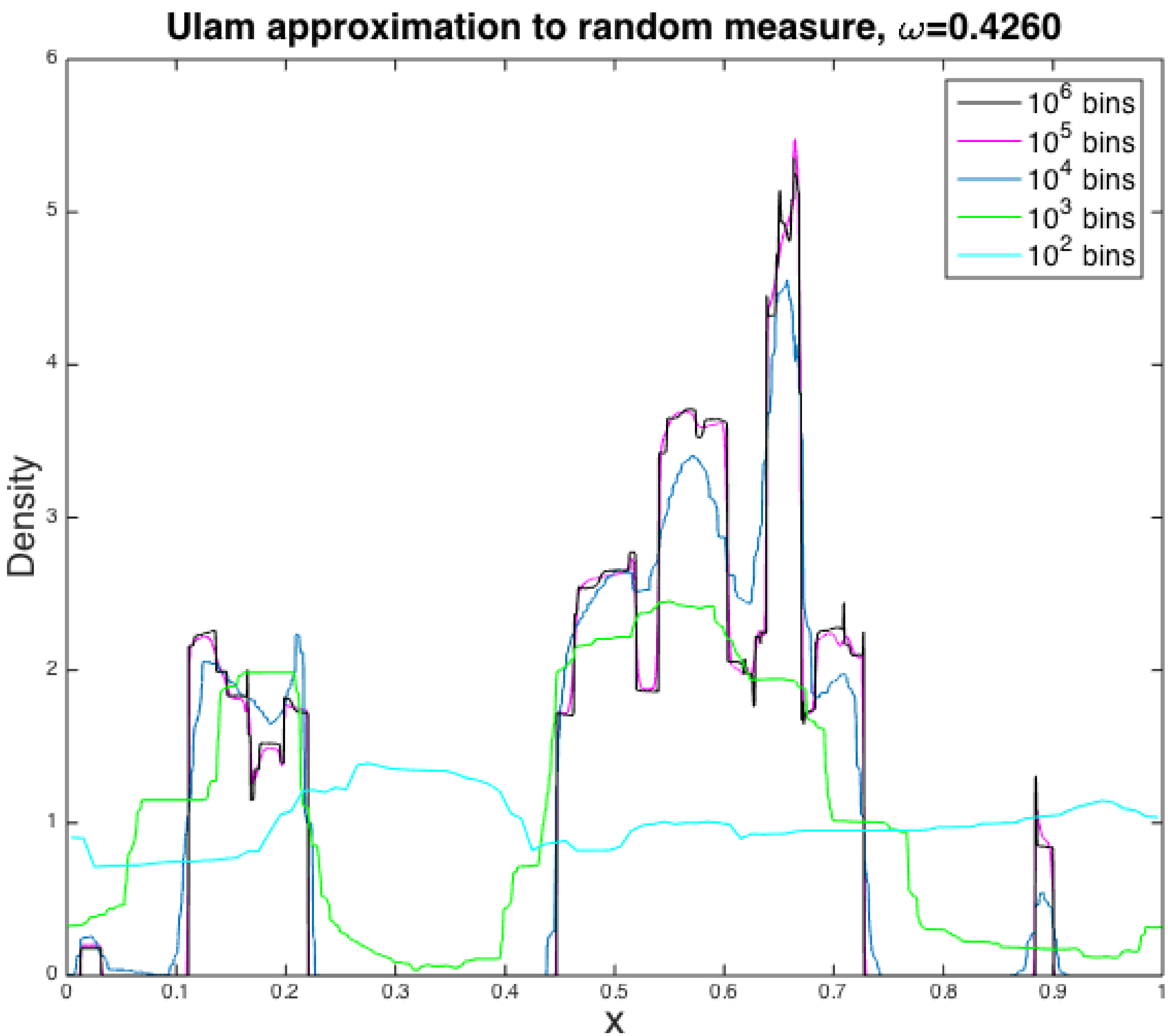
Results are interesting because

1. they are a step towards more general stochastic stability of nonautonomous dynamics
2. from a low dimensional dynamics point of view
   - the actual maps in our examples can have a mix of expansion and contraction
   - mass wanders around, but the statistics are "stable"
   - "physical measure" obtained by averaging φ^*_ω across fibres is accessible, and different to the measure obtained by using $\overline{\mathcal{P}}$

Thankyou!



Approximations to random physical measure
averaged μ_ω (Ulam's method)
histogram (random orbit)
annealed (Ulam's method)
Density
x
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Ulam approximation to random measure, ω=0.4260
Density
x
10^6 bins
10^5 bins
10^4 bins
10^3 bins
10^2 bins
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Abstract

Random dynamical systems are generated by recursively applied sequences of maps, where the choice of map at each timestep is determined by a stochastic process. Such systems are usually formulated as a skew-product, in which the "base" dynamics is autonomous and the "fibre-to-fibre" mappings are determined by the base. An observer watching only the fibres sees non-autonomous dynamics, or "random" orbits. Typical questions of interest relate to the long-term distribution of orbits, mass-transport, rates of mixing and so on, and there are numerous real-world applications.

This talk will introduce the important ideas for studying random dynamics from an ergodic theory viewpoint. Questions of "stochastic stability" can be formulated (and answered) in this way, and much of the theory works as one might expect when the base process is IID. In such cases, insight can even be gained via a transfer operator obtained by averaging over all fibres. When the base process is not IID (for example, an ergodic dynamical system), averaging may yield irrelevant objects, and one must study cocycles of transfer operators and the important dynamical structures on fibres become random variables. This picture will be outlined, and some positive results on accessing the distribution of orbits of certain random interval maps will be given.